# Diophantine Equation $n\left(x^{4}+y^{4}\right)=z^{4}+w^{4}$ 

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#### Abstract

In this paper, we proved that there are infinitely many parametric solutions of $n\left(x^{4}+y^{4}\right)=z^{4}+w^{4}$.

\section*{1. Introduction}


In 2016 Izadi and Nabardi[2] showed $x^{4}+y^{4}=2\left(u^{4}+v^{4}\right)$ has infinitely many integer solutions.
They used a specifc congruent number elliptic curve namely $y^{2}=x^{3}-36 x^{2}$.
In 2019 Janfada and Nabardi[3] showed that a necessary condition for $n$ to have an integral solution for the equation $x^{4}+y^{4}=$ $n\left(u^{4}+v^{4}\right)$ and gave a parametric solution.
They gave the numeric solutions for $n=41,136,313,1028,1201,3281, \cdots$.
' In 2020 Ajai Choudhry [1], Iliya Bluskov and Alexander James showed $\left(x_{1}^{4}+x_{2}^{4}\right)\left(y_{1}^{4}+y_{2}^{4}\right)=z_{1}^{4}+z_{2}^{4}$ has infinitely many Parametric solutions. They gave the numeric solutions for $n=17,257,626,641,706,1921, \cdots$.

Inspired by Choudhry's article, we showed if $n=\left(a^{4}+b^{4}\right) / 2$ then $n\left(x^{4}+y^{4}\right)=z^{4}+w^{4}$ has infinitely many parametric Solutions.
We consider n be an integer. Using four different methods we showed $n\left(x^{4}+y^{4}\right)=z^{4}+w^{4}$ has infinitely many integer solutions. First method is similar as Choudhry's one, second method is using another identity.
CMoreover, we showed the third method through an example $17\left(x^{4}+y^{4}\right)=z^{4}+w^{4}$.
In addition, we showed the fourth method through an example $97\left(x^{4}+y^{4}\right)=z^{4}+w^{4}$.
Finally, we gave the numeric solutions for $n=2,8,17,41,82,97,113,136,137,146,178,193, \cdots$ with $n<1000$.

## 2. First method

Theorem $1\left(x^{4}+y^{4}\right)\left(a^{2}+b^{2}\right)=z^{4}+w^{4}$ has infinitely many parametric solutions, where $(a, b)=\left(\frac{m^{2}-n^{2}}{2}, \frac{m^{2}+n^{2}}{2}\right), m, n$ are arbitrary integer and have same parity.

Proof.
We use Brahmagupta's identity

$$
\left(A^{2}+B^{2}\right)\left(C^{2}+D^{2}\right)=(A C+B D)^{2}+(A D-B C)^{2}
$$

We obtain

$$
\left(x^{4}+y^{4}\right)\left(a^{2}+b^{2}\right)=\left(x^{2} a+y^{2} b\right)^{2}+\left(x^{2} b-y^{2} a\right)^{2}
$$

Hence we have to find the rational solution of $x^{2} a+y^{2} b=u^{2}$ and $x^{2} b-y^{2} a=v^{2}$.
Parametric solution of first equation is $(x, y)=\left(k^{2}-2 m k+4 a-3 m^{2},-k^{2}-2 m k+4 a-m^{2}\right)$ where $a+b=m^{2}$. Substitute it to second equation, we obtain

$$
v^{2}=\left(m^{2}-2 a\right) k^{4}-4 k^{3} m^{3}+\left(4 a m^{2}-2 m^{4}\right) k^{2}+\left(-32 a m^{3}+32 a^{2} m+12 m^{5}\right) k+48 a^{2} m^{2}-32 a^{3}-34 a m^{4}+9 m^{6}
$$

Let $a=\frac{m^{2}-n^{2}}{2}$ and $U=\frac{1}{k}$, then we obtain

$$
\begin{equation*}
V^{2}=\left(5 m^{4} n^{2}+4 n^{6}\right) U^{4}+\left(4 m^{5}+8 m n^{4}\right) U^{3}-2 m^{2} n^{2} U^{2}-4 m^{3} U+n^{2} \tag{1}
\end{equation*}
$$

Since equation (1) has a point $Q(U, V)=(0, n)$, then equation (1) is birationally equivalent to an elliptic curve $E$.

$$
\begin{aligned}
E: Y^{2}-4 m^{3} Y X / n+\left(8 n m^{5}+16 m n^{5}\right) Y & =X^{3}-2 m^{2}\left(n^{4}+2 m^{4}\right) X^{2} /\left(n^{2}\right)+\left(-20 m^{4} n^{4}-16 n^{8}\right) X \\
& +104 m^{6} n^{6}+32 m^{2} n^{10}+80 m^{10} n^{2}
\end{aligned}
$$

Transformation is given,
$U=\frac{2 n^{2} X-4 m^{2} n^{4}-8 m^{6}}{Y n}$
$V=\frac{n^{4} X^{3}-6 n^{6} X^{2} m^{2}-12 X^{2} m^{6} n^{2}+28 n^{8} m^{4} X+32 n^{4} m^{8} X+32 m^{12} X-16 m^{9} Y n+16 n^{9} m Y+16 n^{12} X-104 n^{10} m^{6}-32 n^{14} m^{2}-8}{Y^{2} n^{3}}$
$X=\frac{2 n V+2 n^{2}-4 m^{3} U}{U^{2}}$
$Y=\frac{4 n^{3} V+4 n^{4}-8 n^{2} m^{3} U-4 n^{4} m^{2} U^{2}-8 m^{6} U^{2}}{U^{3} n}$
The point corresponding to point $Q$ is $P(X, Y)=\left(\frac{2 m^{2}\left(n^{4}+2 m^{4}\right)}{n^{2}}, \frac{-16 m\left(-m^{8}+n^{8}\right)}{n^{3}}\right)$.
According to the Nagell-Luts theorem, since the point $P$ is a point of infinite order then we can obtain infinitely many rational points on $E$. Thus we can obtain infinitely many rational solutions of (1).
Therefore $\left(x^{4}+y^{4}\right)\left(a^{2}+b^{2}\right)=z^{4}+w^{4}$ has infinitely many parametric solutions.
The proof is completed.
When $m, n$ have opposite parity, problem links to $h\left(x^{4}+y^{4}\right)=2\left(z^{4}+w^{4}\right)$ where $a^{2}+b^{2}=h / 2$.
Example
$(a, b)=\left(\frac{m^{2}-n^{2}}{2}, \frac{m^{2}+n^{2}}{2}\right)$
$m, n$ are arbitrary integer and have same parity.
$2 Q(U)=\frac{-2 n^{2} m}{n^{4}+m^{4}}$

$$
\begin{aligned}
& x=-n^{4}+4 m^{2} n^{2}+m^{4} \\
& y=n^{4}+4 m^{2} n^{2}-m^{4} \\
& z=\left(3 n^{4}+m^{4}\right) m \\
& w=\left(n^{4}+3 m^{4}\right) n
\end{aligned}
$$

$3 Q(U)=\frac{2 m n^{2}\left(n^{4}+3 m^{4}\right)}{n^{8}-8 m^{4} n^{4}-m^{8}}$

$$
\begin{aligned}
& x=m^{12}+12 n^{2} m^{10}-19 m^{8} n^{4}+40 n^{6} m^{6}+19 m^{4} n^{8}+12 n^{10} m^{2}-n^{12} \\
& y=-m^{12}+12 n^{2} m^{10}+19 m^{8} n^{4}+40 n^{6} m^{6}-19 m^{4} n^{8}+12 n^{10} m^{2}+n^{12} \\
& z=m\left(m^{12}+41 m^{8} n^{4}+27 m^{4} n^{8}-5 n^{12}\right) \\
& w=n\left(-n^{12}-41 m^{4} n^{8}-27 m^{8} n^{4}+5 m^{12}\right)
\end{aligned}
$$

Numerical example: $(m, n)<10$

Table 1: Solutions of $\left(x^{4}+y^{4}\right)\left(a^{2}+b^{2}\right)=z^{4}+w^{4}$

| m | n | $a^{2}+b^{2}$ | x | y | z | w |
| :---: | :--- | :--- | :--- | :--- | :--- | :--- |
| 3 | 1 | 41 | 29 | 11 | 63 | 61 |
| 4 | 2 | 136 | 31 | 1 | 76 | 98 |
| 5 | 1 | 313 | 181 | 131 | 785 | 469 |
| 5 | 3 | 353 | 361 | 89 | 1085 | 1467 |
| 6 | 4 | 776 | 209 | 79 | 774 | 1036 |
| 7 | 1 | 1201 | 649 | 551 | 4207 | 1801 |
| 7 | 3 | 1241 | 1021 | 139 | 4627 | 5463 |
| 7 | 5 | 1513 | 1669 | 781 | 7483 | 9785 |
| 8 | 2 | 2056 | 319 | 191 | 2072 | 1538 |
| 8 | 6 | 2696 | 751 | 401 | 3992 | 5094 |
| 9 | 1 | 3281 | 1721 | 1559 | 14769 | 4921 |
| 9 | 5 | 3593 | 3509 | 541 | 18981 | 25385 |
| 9 | 7 | 4481 | 5009 | 2929 | 30969 | 38647 |

## 3. Second method

Theorem $2\left(x^{4}+y^{4}\right) \frac{m^{4}+1}{2}=z^{4}+w^{4}$ has infinitely many parametric solutions, where $m$ is arbitrary odd number.

Proof.

We use an identity

$$
p(t+1)^{4}+p(t)^{4}=\left(t^{2}+a t+b\right)^{2}+\left(c t^{2}+d t+e\right)^{2}
$$

where

$$
(a, b, c, e, p)=\left(d+2, \frac{1}{2 d}+1,1+d, \frac{1}{2 d}, 1+d+\frac{1}{2 d^{2}}\right)
$$

So, we look for the integer solutions $z^{2}=t^{2}+(d+2) t+\frac{1}{2 d}+1$ and $w^{2}=(1+d) t^{2}+d t+\frac{1}{2 d}$. By parameterizing the first equation and substituting the result to second equation, then we obtain quartic equation.

Let $d=m^{2}-1$ and $U=1 / k$ then

$$
\begin{equation*}
V^{2}=\left(4 m^{6}+5 m^{2}\right) U^{4}+\left(-8 m^{4}-4\right) U^{3}-2 m^{2} U^{2}+4 U+m^{2} \tag{2}
\end{equation*}
$$

This quartic equation is birationally equivalent to an elliptic curve below.

$$
E: Y^{2}-4 Y X / m+\left(8 m+16 m^{5}\right) Y=X^{3}-2\left(m^{4}+2\right) X^{2} /\left(m^{2}\right)+\left(-20 m^{4}-16 m^{8}\right) X+104 m^{6}+32 m^{10}+80 m^{2}
$$

Transformation is given,

$$
\begin{aligned}
& U=\frac{2 m^{2} X-4 m^{4}-8}{Y m} \\
& V=\frac{m^{4} X^{3}-6 m^{6} X^{2}-12 X^{2} m^{2}+28 m^{8} X+32 m^{4} X+32 X+16 Y m-16 m^{9} Y+16 m^{12} X-32 m^{14}-104 m^{10}-80 m^{6}}{Y^{2} m^{3}} \\
& X=\frac{2 m V+2 m^{2}+4 U}{U^{2}} \\
& Y=\frac{4 m^{3} V+4 m^{4}+8 m^{2} U-4 m^{4} U^{2}-8 U^{2}}{U^{3} m}
\end{aligned}
$$

It has a point $P(X, Y)=\left(\frac{2\left(m^{4}+2\right)}{m^{2}}, \frac{-16\left(-1+m^{8}\right)}{m^{3}}\right)$.
According to the Nagell-Luts theorem, since the point $P$ is a point of infinite order then we can obtain infinitely many rational points on $E$.
Thus we can obtain infinitely many rational solutions of (2) using group law.
Therefore $\left(x^{4}+y^{4}\right) \frac{m^{4}+1}{2}=z^{4}+w^{4}$ has infinitely many parametric solutions.
The proof is completed.
When $m$ is even number, problem links to $\left(m^{4}+1\right)\left(x^{4}+y^{4}\right)=2\left(z^{4}+w^{4}\right)$.

## Example

$n=\left(m^{4}+1\right) / 2$
$m$ is odd number.

$$
\begin{aligned}
& x=m^{4}+4 m^{2}-1 \\
& y=m^{4}-4 m^{2}-1 \\
& z=3 m^{4}+1 \\
& w=\left(m^{4}+3\right) m
\end{aligned}
$$

$$
\begin{aligned}
& x=m^{12}+12 m^{10}-19 m^{8}+40 m^{6}+19 m^{4}+12 m^{2}-1 \\
& y=m^{12}-12 m^{10}-19 m^{8}-40 m^{6}+19 m^{4}-12 m^{2}-1 \\
& z=5 m^{12}-27 m^{8}-41 m^{4}-1 \\
& w=m\left(m^{12}+41 m^{8}+27 m^{4}-5\right)
\end{aligned}
$$

## 4. Example for $41\left(x^{4}+y^{4}\right)=z^{4}+w^{4}$

We show a numerical example of first method.
Let $(a, b, m)=(4,5,3)$ then $\left(x^{4}+y^{4}\right)\left(a^{2}+b^{2}\right)=z^{4}+w^{4}$ is reduced to

$$
V^{2}=U^{4}-108 U^{3}-18 U^{2}+996 U+409
$$

Quartic equation is birationally equivalent to an elliptic curve $E$.

$$
E: Y^{2}+X Y+Y=X^{3}-X^{2}-27 X+26
$$

Transformation is given,

$$
\begin{aligned}
U & =\frac{4 Y+29 X-10}{X-46} \\
V & =\frac{16 X^{3}-2208 X^{2}-3392 X+13744-9840 Y}{(X-46)^{2}} \\
X & =\frac{V+U^{2}-54 U+5}{32} \\
Y & =\frac{U V+U^{3}-83 U^{2}+99 U-29 V+175}{128}
\end{aligned}
$$

The rank of elliptic curve $E$ is 1 and has generator $P(X, Y)=(6,-11)$ using SAGE. We can find infinitely many rational points on the curve E using the group law. Thus the multiples $\mathrm{nP}, \mathrm{n}=2,3, \ldots$ give infinitely many points as follows.

Table 2: Solutions of $41\left(x^{4}+y^{4}\right)=z^{4}+w^{4}$

| nP | x | y | z | w |
| :--- | :--- | :--- | :--- | :--- |
| $2 P$ | 29 | 11 | 63 | 61 |
| $3 P$ | 17909 | 5149 | 37623 | 38699 |
| $4 P$ | 229422601 | 214213319 | 663306603 | 282177719 |
| $5 P$ | 81840455152441 | -86237007592439 | 252933880274523 | 61172008172039 |

## 5. Example for $17\left(x^{4}+y^{4}\right)=z^{4}+w^{4}$

The previous methods can't give the rational solution of $17\left(x^{4}+y^{4}\right)=z^{4}+w^{4}$. Hence we show the third method of giving the rational solution to the equation $\left(x^{4}+y^{4}\right)\left(a^{2}+b^{2}\right)=z^{4}+w^{4}$. We consider the simultaneous equation as follows.

$$
\left\{\begin{array}{l}
z^{2}-w^{2}-5 x^{2}-3 y^{2}=2 t\left(4 x^{2}+w^{2}-y^{2}\right) \\
\left(w^{2}+z^{2}+5 x^{2}+3 y^{2}\right) t=-\left(x^{2}+w^{2}+4 y^{2}\right)
\end{array}\right.
$$

Eliminating $z^{2}$, we get

$$
\begin{equation*}
\left(1+10 t+8 t^{2}\right) x^{2}+\left(4+6 t-2 t^{2}\right) y^{2}+\left(2 t^{2}+1+2 t\right) w^{2}=0 \tag{3}
\end{equation*}
$$

Parametric solution of $(3)$ is $(x, y, w)=\left(2(k-2)(k-3),(k-1)(3 k-7),-\left(25+5 k^{2}-22 k\right)\right)$.

$$
\begin{aligned}
z^{2} & =\frac{-w^{2}+17 y^{2}}{4} \\
& =\frac{4\left(2 k^{2}-5 k+1\right)\left(4 k^{2}-15 k+13\right)}{\left(-5+k^{2}\right)^{2}}
\end{aligned}
$$

Hence cosider the quartic equation (4)

$$
\begin{equation*}
V^{2}=8 U^{4}-50 U^{3}+105 U^{2}-80 U+13 \tag{4}
\end{equation*}
$$

Quartic equation (4) has a rational point $Q(U, V)=(2,1)$, then this quartic equation is birationally equivalent to an elliptic curve below.
$E: Y^{2}=X^{3}-91 X+330$ with rank is 1 and generator $=(7,-6)$.
Transformation is given,

$$
\begin{aligned}
U & =\frac{6 X-36+2 Y}{Y+2 X-12} \\
V & =\frac{-18 X^{2}-114+X^{3}+91 X}{(Y+2 X-12)^{2}} \\
X & =\frac{2 V+6-U^{2}}{U^{2}-4 U+4} \\
Y & =\frac{12 V-108+180 U-90 U^{2}-4 V U+14 U^{3}}{U^{3}-6 U^{2}+12 U-8}
\end{aligned}
$$

The rank of elliptic curve $E$ is 1 and has generator $P(X, Y)=(7,-6)$ using SAGE. We can find infinitely many rational points on the curve E using the group law. Thus the multiples $\mathrm{nP}, \mathrm{n}=2,3, \ldots$ give infinitely many points as follows.

Table 3: Solutions of $17\left(x^{4}+y^{4}\right)=z^{4}+w^{4}$

| nP | x | y | z | w |
| :--- | :--- | :--- | :--- | :--- |
| $2 P$ | 3120 | 1921 | 2242 | 6529 |
| $3 P$ | 18418554 | 88538885 | 176117272 | 95896333 |
| $4 P$ | 87733253643360 | 108376421998081 | 198203611434238 | 206237591201281 |
| $5 P$ | 12509563104278834954874 | 6446124521923428875525 | 3117838409641509334568 | 25836199364300466735373 |

## 6. Example for $97\left(x^{4}+y^{4}\right)=z^{4}+w^{4}$

The previous methods can't give the rational solution of $97\left(x^{4}+y^{4}\right)=z^{4}+w^{4}$. Hence we show the fourth method of giving the rational solution to the equation $\left(x^{4}+y^{4}\right)\left(a^{2}+b^{2}\right)=z^{4}+w^{4}$. According to Richmond's theorem [4], existence of solution for diophantine equation $a x^{4}+b y^{4}+c z^{4}+d w^{4}=0$ are known if $a b c d$ is square number. He proved other solution is derived from a known solution. Repeting this process, we can obtain infinitely many integer solutions. We use a known solution $(x, y, z, w)=(112,71,10,37)$ obtained by brute force as follows.
Let $x=p t+112, y=q t+71, z=r t+10, w=t+37$.
We obtain

$$
\begin{aligned}
& \left(q^{4}+p^{4}-97 r^{4}-97\right) t^{4}+\left(-3880 r^{3}+284 q^{3}-14356+448 p^{3}\right) t^{3} \\
& +\left(-796758-58200 r^{2}+30246 q^{2}+75264 p^{2}\right) t^{2}+(-19653364+5619712 p-388000 r+1431644 q) t=0
\end{aligned}
$$

In order to set the coefficient of $t$ and $t^{2}$ to zero, we obtain

$$
\begin{aligned}
q & =\frac{-135744 p}{42103}+\frac{491693}{42103} \\
r & =\frac{-22422}{2965}+\frac{7672 p}{2965}
\end{aligned}
$$

Hence we obtain

$$
t=\frac{-1238771307200}{68835707869 p-174887242544}
$$

Take $\mathrm{p}=1$ then $(q, r)=\left(\frac{355949}{42103}, \frac{-2950}{593}\right)$.
Finally, we obtain $(x, y, z, w)=(174887242544,240033770927,68026751110,68835707869)$.
Thus we can obtain other integer solution by using this new solution as a known solution. According to Richmond's theorem, all solutions of Table 4 have infinitely many integer solutions.

Smallest: $\operatorname{Min} n\left(x^{4}+y^{4}\right)$
Brute force search range: $n<1000,(x, y, z, w)<50000$ $n \equiv 1,2,8,9 \bmod 16$ is to be searched.

Table 4: Solutions of $n\left(x^{4}+y^{4}\right)=\left(z^{4}+w^{4}\right)$
$\mathrm{n} \quad \mathrm{x} \quad \mathrm{y} \quad \mathrm{z} \quad \mathrm{w}$

| n | x | y | z | w |
| ---: | ---: | ---: | ---: | ---: |
| 2 | 7 | 20 | 21 | 19 |
| 8 | 19 | 21 | 40 | 14 |
| 17 | 5 | 6 | 13 | 8 |
| 41 | 1 | 1 | 3 | 1 |
| 82 | 219 | 320 | 1011 | 247 |
| 97 | 10 | 37 | 112 | 71 |
| 113 | 1 | 2 | 6 | 5 |
| 136 | 1 | 1 | 4 | 2 |
| 137 | 29 | 5 | 99 | 31 |
| 146 | 1 | 2 | 7 | 3 |
| 178 | 1 | 2 | 7 | 5 |
| 193 | 18 | 43 | 159 | 80 |
| 226 | 1 | 8 | 31 | 7 |
| 241 | 1 | 2 | 8 | 1 |
| 257 | 4 | 15 | 52 | 49 |
| 313 | 1 | 1 | 5 | 1 |
| 328 | 7 | 3 | 30 | 8 |
| 337 | 15 | 34 | 147 | 26 |
| 353 | 1 | 1 | 5 | 3 |
| 386 | 1 | 2 | 9 | 1 |
| 401 | 1 | 2 | 9 | 4 |
| 433 | 8 | 13 | 53 | 50 |
| 466 | 2 | 43 | 181 | 151 |
| 482 | 5 | 6 | 31 | 7 |
| 521 | 9 | 1 | 43 | 1 |
| 562 | 10 | 11 | 61 | 7 |
| 577 | 175 | 188 | 929 | 848 |
| 578 | 24 | 37 | 187 | 85 |
| 593 | 1 | 2 | 10 | 3 |
| 626 | 16 | 29 | 141 | 97 |
| 641 | 16 | 19 | 97 | 78 |
| 673 | 161 | 26 | 820 | 139 |
| 706 | 17 | 28 | 149 | 13 |
| 712 | 15 | 13 | 86 | 36 |
| 761 | 7 | 3 | 37 | 11 |
| 776 | 1 | 1 | 6 | 4 |
| 802 | 1 | 10 | 53 | 19 |
| 857 | 7 | 5 | 39 | 23 |
| 866 | 1 | 2 | 11 | 3 |
| 881 | 7 | 31 | 169 | 9 |
| 898 | 1 | 2 | 11 | 5 |
| 953 | 2041 | 2021 | 12975 | 7999 |
| 977 | 3 | 10 | 56 | 11 |
|  |  |  |  |  |
|  |  |  |  |  |

## 7. Final remarks

There is an interesting relationship between $8\left(19^{4}+21^{4}\right)=14^{4}+40^{4}$ and $2\left(7^{4}+20^{4}\right)=19^{4}+21^{4}$
Multiply both sides of $2\left(7^{4}+20^{4}\right)=19^{4}+21^{4}$ by $2^{3}$, then we obtain $14^{4}+40^{4}=8\left(19^{4}+21^{4}\right)$.
Thus an equation $n\left(x^{4}+y^{4}\right)=z^{4}+w^{4}$ can be transform $(n x)^{4}+(n y)^{4}=n^{3}\left(z^{4}+w^{4}\right)$.
We could not find a solution of $n\left(x^{4}+y^{4}\right)=z^{4}+w^{4}$ for $n=34,40,50,65,73,89$ whre $n<100$. They are not the form $\left(m^{4}+n^{4}\right) / 2$. I don't know if they don't have a solution to begin with, or if they have a large solution.

## References

[1] Ajai Choudhry, Iliya Bluskov and Alexander James, A quartic diophantine equation inspired by Brahmagupta's identity, 2020, arXiv.org
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[3] Ali S. Janfada, Kamran Nabardi, ON DIOPHANTINE EQUATION $x^{4}+y^{4}=n\left(u^{4}+v^{4}\right)$, December 2019, Mathematica Slovaca 69(6)
[4] H. W. Richmond, On the Diophantine equation $F=a x^{4}+b y^{4}+c z^{4}+d w^{4}$, the product $a b c d$ being square number, J. Lond. Math. Soc., 19 (1944)

